MATH 504 HOMEWORK 5

Due Monday, April 12.

Problem 1. *Let M be a countable transitive model of ZFC, λ be a regular cardinal and let $Add(\omega, \lambda)$ be the poset of all partial functions from $\lambda \times \omega$ to $\{0, 1\}$ with finite domain. Let G be a generic filter over M. Define $f^* : \lambda \times \omega \to \{0, 1\}$ to be $f^* = \bigcup G$ and for all $\alpha < \lambda$, let $f_\alpha : \omega \to \{0, 1\}$ be $f_\alpha(n) = f^*(\alpha, n)$. Prove that*

1. *f^* is a total function with domain $\lambda \times \omega$.*
2. *For each $\alpha < \beta < \lambda$, $f_\alpha \neq f_\beta$.*
3. *For each $\alpha < \lambda$, $f_\alpha \notin M$.*

Problem 2. *Let M be a transitive model of ZFC and $\mathbb{P} \in M$ be a poset. Suppose that $p \in \mathbb{P}$ is such that $p \Vdash \dot{f} : \lambda \to \tau$ is a function.*

1. *Show that for every $\alpha < \lambda$, the set $\{q \mid \exists\gamma < \tau(q \Vdash \dot{f}(\alpha) = \gamma)\}$ is dense below p.*
2. *Let $B = \{\gamma < \tau \mid (\exists q \leq p)(\exists\alpha < \lambda)(q \Vdash \dot{f}(\alpha) = \gamma)\}$. Show that if $\sup(B) < \tau$, then $p \Vdash \operatorname{ran}(\dot{f})$ is bounded in τ.*

Problem 3. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are two posets and $i : \mathbb{P} \to \mathbb{P}$ is such that:*

1. *$i(1_\mathbb{P}) = 1_\mathbb{Q}$;*
2. *If $p' \leq p$, then $i(p') \leq i(p)$;*
3. *For all $p_1, p_2 \in \mathbb{P}$, $p_1 \perp p_2$ iff $i(p_1) \perp i(p_2)$;*
4. *If A is a maximal antichain of $\mathbb{P}$, then $i"A := \{i(p) \mid p \in A\}$ is a maximal antichain in $\mathbb{Q}$.*

Suppose also that H is $\mathbb{Q}$-generic. Show that $G := \{p \in \mathbb{P} \mid i(p) \in H\}$ is $\mathbb{P}$-generic and that $V[G] \subset V[H]$, where V is the ground model.

Remark: an embedding as above is called **a complete embedding**,

Problem 4. *Suppose that for all n, $2^{\aleph_n} = \aleph_{\omega+1}$. Show that $2^{\aleph_\omega} = \aleph_{\omega+1}$. Hint: For each $A \subset \aleph_\omega$, define $A_n := A \cap \aleph_n$. Consider the map $A \mapsto \langle A_n \mid n < \omega\rangle$.*

Problem 5. *Suppose that $\mathbb{P}$ is a poset, $A \subset \mathbb{P}$ is a maximal antichain, $\phi(x)$ is a formula, and $\langle\tau_p \mid p \in A\rangle$ are $\mathbb{P}$ names, such that for all $p \in A$, $p \Vdash \phi(\tau_p)$. Show that there is a $\mathbb{P}$ name τ, such that $1_\mathbb{P} \Vdash \phi(\tau)$.*